Systems of Equations and Inequalities

Precalculus: Chapter 8

- This Slideshow was developed to accompany the textbook
  - *Precalculus*
  - *By Richard Wright*
  - *https://www.andrews.edu/~rwright/Precalculus-RLW/Text/TOC.html*
- Some examples and diagrams are taken from the textbook.

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8-01 Nonlinear and Linear Systems

In this section, you will:

- Solve a system of equations by graphing.
- Solve a system of equations by substitution.

8-01 Nonlinear and Linear Systems

- System of Equations
  - Several equations with the same solution
- Substitution
  1. Solve one equation for a variable
  2. Substitute this expression into the other equation
  3. Solve the new equation
  4. Substitute the solution back into the 1st equation and solve
  5. Check your answers!

8-01 Nonlinear and Linear Systems

- Solve $\begin{cases} -2x + y = 5 \\ x^2 + 3x - y = 1 \end{cases}$

$$\begin{cases} -2x + y = 5 \\ x^2 + 3x - y = 1 \end{cases}$$

$$y = 5 + 2x$$

$$x^2 + 3x - (5 + 2x) = 1$$

$$x^2 + x - 5 = 1$$

$$x^2 + x - 6 = 0$$

$$(x - 2)(x + 3) = 0$$

$$x - 2 = 0 \qquad x + 3 = 0$$

$$x = 2 \qquad x = -3$$

X=2

$$y = 5 + 2(2) = 9$$

X=3

$$y = 5 + 2(-3) = -1$$

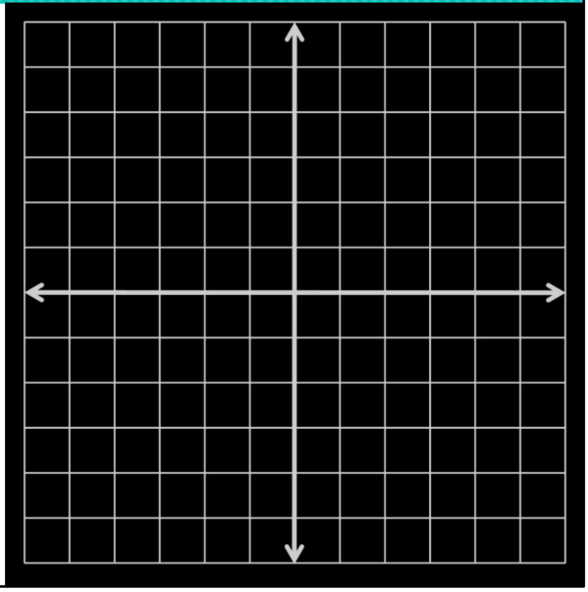
$$(2, 9), (-3, -1)$$

8-01 Nonlinear and Linear Systems

- Graphical Method
  - Graph both equations on same coordinate plane
  - The points of intersection are the solutions.

8-01 Nonlinear and Linear Systems

- Solve graphically
- $\begin{cases} x^2 - y = 5 \\ -x + y = -3 \end{cases}$



Make both equations equal to y

$$\begin{cases} y = x^2 - 5 \\ y = x - 3 \end{cases}$$

Graph

(−1, −4), (2, −1)

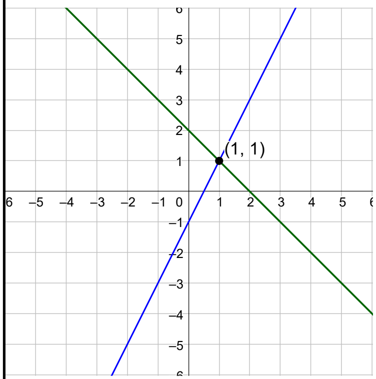
8-02 Two-Variable Linear Systems

In this section, you will:

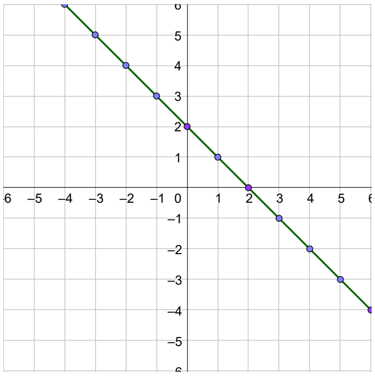
- Solve two-variable linear systems using elimination.
- Classify types of solutions.

8-02 Two-Variable Linear Systems

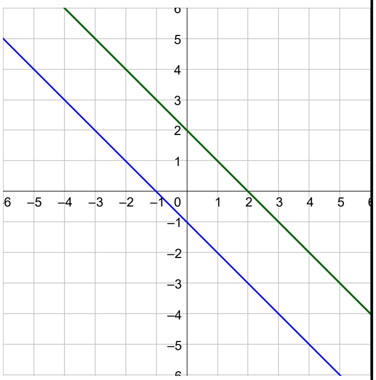
- Three possibilities for solutions



1 Solution: Consistent and Independent



Infinitely Many Solutions: Consistent and Dependent



No Solution: Inconsistent

8-02 Two-Variable Linear Systems

Solving Linear Equations by Elimination

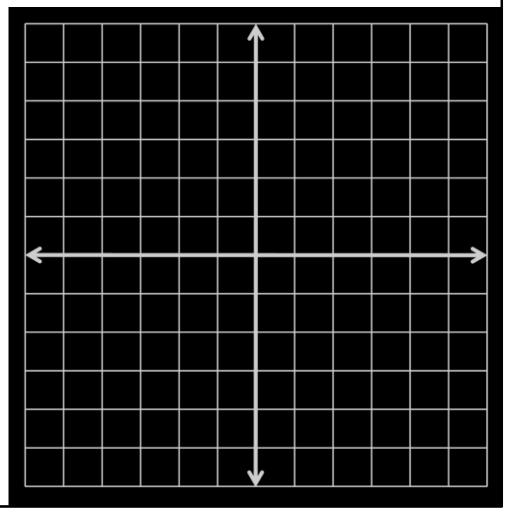
1. Write the equations in columns ($ax + by = c$)
2. Obtain coefficients of one variable that differ only in sign by multiplying the equations by constants.
3. Add the equations and solve the resulting equation. (A variable will be eliminated.)
4. Back-substitute the answer into either original equation and solve.
5. Check your solution.

8-02 Two-Variable Linear Systems

- If all the variables are eliminated and the result is
  - True (like 0 = 0), there are infinitely many solutions
  - False (like 0 = 9), there are no solutions

8-02 Two-Variable Linear Systems

- Solve $\begin{cases} 4x + y = -3 \\ x - 3y = 9 \end{cases}$



$$\begin{cases} 4x + y = -3 \\ x - 3y = 9 \end{cases}$$

$$12x + 3y = -9$$
$$x - 3y = 9$$

$$13x = 0$$
$$x = 0$$

$$4(0) + y = -3$$
$$y = -3$$

(0, -3)

8-03 Multivariable Linear Systems

In this section, you will:

- Use elementary row operations.
- Solve systems of linear equations by putting them in row-echelon form.
- Write the answer to a three-variable system of equations with many solutions.

8-03 Multivariable Linear Systems

- **Row-Echelon Form**
  - The first nonzero term in each equation has a coefficient of 1.
  - All terms under the leading 1 are zero producing an inverted pyramid shape.
  - Any equations that are all zeros are at the bottom.

- $$\begin{cases} 1x + y + 3z = 3 \\ \quad\;\; 1y + 5z = 10 \\ \qquad\qquad 1z = 7 \end{cases}$$

8-03 Multivariable Linear Systems

- **Elementary Row Operations**
- The following operations are allowed in systems of equations and produce equivalent systems.
  - Interchange two equations
  - Multiply one equation by a nonzero constant
  - Add a multiple of one equation to another equation and replace the latter equation

8-03 Multivariable Linear Systems

- Solve

$$\begin{cases} x + y + z = 3 \\ 2x - y + 3z = 16 \\ x - 2y - z = 1 \end{cases}$$

$$-2\begin{cases} x + y + z = 3 \\ 2x - y + 3z = 16 \\ x - 2y - z = 1 \end{cases}$$

$$-1\begin{cases} x + y + z = 3 \\ -3y + z = 10 \\ x - 2y - z = 1 \end{cases}$$

$$-1\begin{cases} x + y + z = 3 \\ -3y + z = 10 \\ -3y - 2z = -2 \end{cases}$$

$$\begin{matrix} \\ -1/3 \\ -1/3 \end{matrix}\begin{cases} x + y + z = 3 \\ -3y + z = 10 \\ -3z = -12 \end{cases}$$

$$\begin{cases} x + y + z = 3 \\ y - \dfrac{1}{3z} = -\dfrac{10}{3} \\ z = 4 \end{cases}$$

$$y = -\frac{10}{3} + \frac{1}{3}(4) = -2$$
$$x = 3 - (-2) - 4 = 1$$

(1, -2, 4)

8-03 Multivariable Linear Systems

- Solve

$$\begin{cases} x + 2y \quad - 7z = -4 \\ 2x + 3y \quad + z = 5 \\ 3x + 7y - 36z = -25 \end{cases}$$

$$-2\begin{cases} x + 2y - 7z = -4 \\ 2x + 3y + z = 5 \\ 3x + 7y - 36z = -25 \end{cases}$$

$$-3\begin{cases} x + 2y - 7z = -4 \\ -y + 15z = 13 \\ 3x + 7y - 36z = -25 \end{cases}$$

$$\begin{cases} x + 2y - 7z = -4 \\ -y + 15z = 13 \\ y - 15z = -13 \end{cases}$$

$$-1\begin{cases} x + 2y - 7z = -4 \\ -y + 15z = 13 \\ 0 = 0 \end{cases}$$

Many solutions
Let z=a

$$y = 15z - 13$$
$$y = 15a - 13$$

$$x + 2(15a - 13) - 7a = -4$$
$$x + 30a - 26 - 7a = -4$$
$$x = -23z + 22$$

(-23a+22, 15a-13, a)

8-03 Multivariable Linear Systems

- Solve

$$\begin{cases} x - y + 4z = 3 \\ 4x - z = 0 \end{cases}$$

$$-4 \begin{cases} x - y + 4z = 3 \\ -z = 0 \end{cases}$$

$$1/4 \begin{cases} x - y + 4z = 3 \\ 4y - 17z = -12 \end{cases}$$

$$\begin{cases} x - y + 4z = 3 \\ y - \frac{17}{4}z = -3 \end{cases}$$

Let z=4a (so no fractions)

$$y - \frac{17}{4}(4a) = -3$$
$$y = 17a - 3$$

$$x - (17a - 3) + 4(4a) = 3$$
$$x - 17a + 3 + 16a = 3$$
$$x = a$$

(a, 17a-3, 4a)

In this section, you will:

Decompose rational expressions into partial fractions with:

- Distinct linear factors
- Repeated linear factors
- Distinct quadratic factors
- Repeated quadratic factors

8-04 Partial Fractions

8-04 Partial Fractions

- To split a rational function into smaller fractions
  - Like $\frac{5}{12} = \frac{1}{6} + \frac{1}{4}$
  - $\frac{x+8}{x^2+6x+8} = \frac{?}{x+2} + \frac{?}{x+4}$

8-04 Partial Fractions

- **To Find Partial Fractions**
- Factor the denominator.
- For each linear factor of the denominator are in the form

$$\frac{A}{px+q}+\frac{B}{(px+q)^2}+\cdots$$

- For each quadratic factor of the denominator are in the form

$$\frac{Ax+B}{ax^2+bx+c}+\frac{Cx+D}{(ax^2+bx+c)^2}+\cdots$$

- Solve for A, B, C, etc.
  - Multiply by the LCD
  - Choose convenient values of x to find A, B, C, etc.
  - Or create a system of linear equations based on the coefficients of x.

8-04 Partial Fractions

- Find the partial fractions

$$\frac{x+8}{x^2+6x+8} =$$

$$\frac{x+8}{(x+2)(x+4)} = \frac{\mathrm{A}}{\mathrm{x}+2} + \frac{\mathrm{B}}{\mathrm{x}+4}$$

Multiply by LCD

$$x + 8 = A(x+4) + B(x+2)$$

Shortcut: Use convenient x-values

Let x=-2: $-2 + 8 = A(-2+4) + B(-2+2)$

$$6 = 2A$$

$$A = 3$$

Let x=-4: $-4 + 8 = A(-4+4) + B(-4+2)$

$$4 = -2B$$

$$B = -2$$

$$\frac{3}{x+2} - \frac{2}{x+4}$$

8-04 Partial Fractions

- $\frac{3x^2-x+5}{x^3-2x^2+x}$

$$\frac{3x^2-x+5}{x(x-1)(x-1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Multiply by LCD

$$3x^2 - x + 5 = A(x-1)^2 + Bx(x-1) + Cx$$

Convenient values don't work. Multiply out and combine like terms of x.

$$3x^2 - x + 5 = Ax^2 - 2Ax + A + Bx^2 - Bx + Cx$$
$$3x^2 - x + 5 = (A+b)x^2 + (-2A - B + C)x + A$$

Make equations out of like terms of x.

$$\begin{cases} A + B = 3 \\ -2A - B + C = -1 \\ A = 5 \end{cases}$$

Top equation

$$5 + B = 3$$
$$B = -2$$

Middle equation

$$-2(5) - (-2) + C = 1$$
$$C = 7$$
$$\frac{5}{x} + \frac{-2}{x-1} + \frac{7}{(x-1)^2}$$

8-04 Partial Fractions

- $\frac{6x^3+16x}{(x^2+3)^2}$

$$\frac{6x^3+16x}{(x^2+3)^2} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{(x^2+3)^2}$$

Multiply by LCD

$$6x^3+16x = (Ax+B)(x^2+3) + (Cx+D)$$

Convenient values don't work, so multiply out and gather like terms of x

$$6x^3+16x = Ax^3+Bx^2+3Ax+3B+Cx+D$$
$$6x^3+16x = Ax^3+Bx^2+(3A+C)x+(3B+D)$$

Make equations out of the like terms.

$$\begin{cases} A = 6 \\ B = 0 \\ 3A + C = 16 \\ 3B + D = 0 \end{cases}$$

Third equation

$$3(6) + C = 16$$
$$C = -2$$

Fourth equation

$$3(0) + D = 0$$
$$D = 0$$

$$\frac{6x}{x^2+3}+\frac{-2x}{(x^2+3)^2}$$

8-05 Systems of Inequalities

In this section, you will:

- Graph systems of equations.
- Find the vertices of a graph of a system of equations.

8-05 Systems of Inequalities

- **Solve Systems of Inequalities**
  - Graph all the inequalities on the same coordinate plane.
  - Find the intersection of the shaded areas.

8-05 Systems of Inequalities

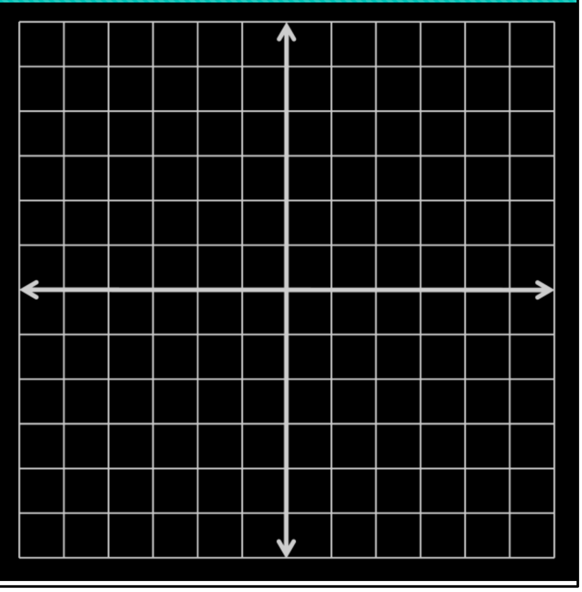
- **Graph an Inequality**

1. Pretend the inequality sign is = and graph the line.
2. Decide if the line is solid or dotted
   - Solid if ≤, =, or ≥
   - Dotted if < or >
3. Shade
   - Pick a test point not on the line.
     - Plug the test point into the inequality
     - If this results in a true statement, then shade the side of the graph with the test point.
     - If the result is **not** a true statement, then shade the other side of the graph.
   - OR if solved for *y*
     - *y* > shade above the line.
     - *y* < shade below the line.

8-05 Systems of Inequalities

- Solve

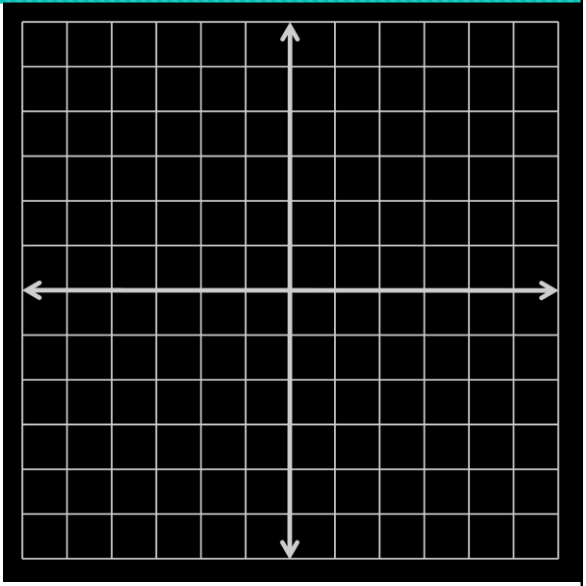
$$\begin{cases} x + y \geq 1 \\ -x + y \geq 1 \\ y \leq 2 \end{cases}$$



8-05 Systems of Inequalities

- Solve

$$\begin{cases} y \geq x^2 \\ y > x + 2 \end{cases}$$



8-06 Linear Programming

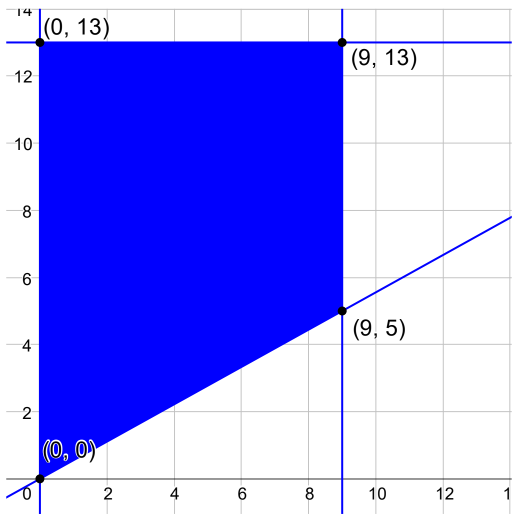
In this section, you will:

- Use linear programming to maximize or minimize a function.

8-06 Linear Programming

- Optimization strategy
  - Maximize or minimize
- Objective function to optimize
- Constraints – system of inequalities

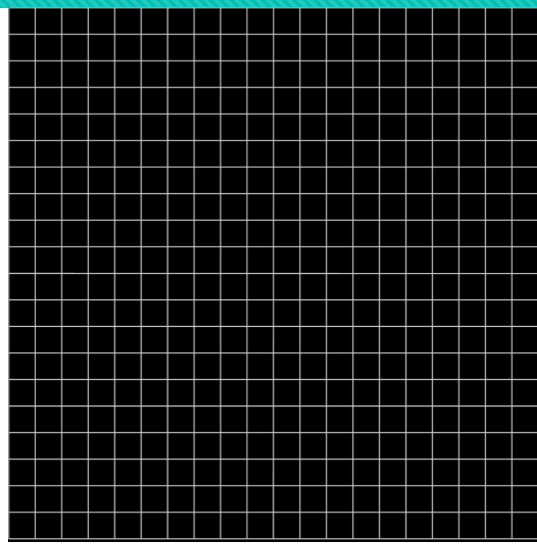
1. Graph constraints and find vertices of solution
   - Max or min will be at vertex
2. Plug vertices into objective function to find min or max



8-06 Linear Programming

- Find the minimum value of $z = 4x + 6y$ subject to

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ x + y \geq 2 \\ y \leq 4 \\ x \leq 5 \end{cases}$$



Test vertices
(2, 0): $z = 4(2) + 6(0) = 8$
(5, 0): $z = 4(5) + 6(0) = 20$
(5, 4): $z = 4(5) + 6(4) = 44$
(0, 4): $z = 4(0) + 6(4) = 24$
(0, 2): $z = 4(0) + 6(2) = 12$

Minimum is 8 at (2, 0)